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Positiveness of Mass and the Strong Energy Condition*

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Abstract

An important question in general relativity is, what conditions are sufficient to guarantee that the mass of a bounded system be positive? We approach this problem for static nonvacuum systems with the help of a formula for the gravitational mass (a generalization of one given earlier by Tolman) which separates the contributions of the singularities from those of the matter fields. For a singularity-free system, if the matter fields obey the strong energy condition familiar from the "singularity theorems," then the mass will be positive. For systems with matter not obeying the strong energy condition, a much weakened energy condition is still sufficient to guarantee positive mass. We illustrate both of the cases with concrete examples.

1. *Introduction*

An important question in general relativity is, what conditions are sufficient to guarantee that the mass of a bounded system be positive? This problem has been considered extensively for vacuum systems (Brill and Deser, 1968a; Brill, Deser, and Fadeev, 1968), but has received scanty attention for the nonvacuum case (Brill and Deser, 1968b). Here we present a simple approach to the problem for the case of static nonvacuum systems. We first derive a formula for the gravitational mass (a generalization of an earlier one by Tolman) which separates the contributions of singularities from those of the matter fields. It follows from it that for a singularity-free system, satisfaction of the strong energy condition familiar from the "singularity theorems" (Penrose and Hawking, 1970; Geroch, 1966) is sufficient to guarantee positive mass. We illustrate with the example of a system composed of a massive vector field with sources. Some forms of matter may disobey the strong energy condition. For these a weakened form of the energy condition may still prove sufficient to guarantee positive mass. We illustrate with the example of a system composed of a massive scalar field with sources.

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2. *The Mass of a Bounded Static System*

For any static system the line element can be written as

$$ds^2 = -h^2(dx^0)^2 + \gamma_{ij}\, dx^i\, dx^j \tag{2.1}$$

where $h_{,0} = \gamma_{ij,0} = 0$. The time–time component of the Ricci tensor is (Weyl, 1952)

$$R_0{}^0 = -(-g)^{-1/2}(\gamma^{1/2}\gamma^{ij}h_{,j})_{,i} \tag{2.2}$$

where $\gamma^{ij}\gamma_{jk} = \delta_k{}^i$. Integrating $R_0{}^0(-g)^{1/2}$ over all 3-space away from its singularities and using Gauss's theorem we get

$$-\int R_0{}^0(-g)^{1/2}\, d^3x = \int_I \gamma^{ij}h_{,j}\, dS_i - \int_S \gamma^{ij}h_{,j}\, dS_i \tag{2.3}$$

where the surface integrals at infinity (I) and those surrounding the singularities (S) are taken with the usual 2-surface element $dS_i = \frac{1}{2}\gamma^{1/2}\epsilon_{ijk}\, dx^j\, dx^k$. In both integrals the normals point outward.

Let us assume that our space-time is asymptotically flat; we may then choose the space coordinates so that asymptotically they reduce to spherical polar coordinates r, θ, and ϕ. We know that asymptotically $h \to 1 - M/r$ where M is the gravitational mass of the system. It is then easy to see that the integral over infinity in (2.3) is just $4\pi M$. Since by Einstein's equations $R_0{}^0 = 8\pi(T_0{}^0 - \frac{1}{2}T)$, (2.3) implies

$$M = \frac{1}{4\pi}\int_S \gamma^{ij}h_{,j}\, dS_i - 2\int (T_0{}^0 - \tfrac{1}{2}T)(-g)^{1/2}\, d^3x \tag{2.4}$$

The special case of this formula without the singularity term was obtained earlier by Tolman (1934) by a different method. The surface integral in (2.4) can in fact be taken over any closed 2-surface which encloses the singularities, provided the volume integral extends only outside this surface.

For a vacuum space-time M is given merely by the first integral in (2.4); the mass is determined directly by the behavior of the metric near the singularity. The same happens to be true for a space-time containing only a massless scalar field ϕ; then

$$T_{\alpha\beta} = \phi_{,\alpha}\phi_{,\beta} - \tfrac{1}{2}g_{\alpha\beta}\phi_{,\gamma}\phi_{,}{}^{\gamma} \tag{2.5}$$

and since $\phi_{,0} = 0$, $T_0{}^0 - \frac{1}{2}T = 0$.

3. *Systems Obeying the Strong Energy Condition*

We now specialize to nonvacuum systems devoid of singularities; this means in general that the fields present will have sources. We shall assume that $h^2 > 0$ everywhere, thus excluding the case with an horizon (black hole) from consideration. We now define a unit timelike vector by

$$u^\alpha = h^{-1}\delta_0{}^\alpha \tag{3.1}$$

In terms of it (2.4) can be written as

$$M = 2 \int T^{*}_{\alpha\beta} u^{\alpha} u^{\beta} (-g)^{1/2}\, d^3x \tag{3.2}$$

where $T^{*}_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T$. We recall that in the "singularity theorems" (Penrose and Hawking, 1970; Geroch, 1966) the strong energy condition for an arbitrary unit timelike vector u^{α}

$$T^{*}_{\alpha\beta} u^{\alpha} u^{\beta} \geqslant 0 \tag{3.3}$$

plays an important role. It is evident from (3.2) that if the matter in a system obeys the strong energy condition, then the mass must be positive.

As an example, we consider a massive vector field B_{α} of Compton length μ^{-1} coupled to point sources with strength e_i; the appropriate action functional for the field and coupling is

$$S = -\int [H_{\alpha\beta} H^{\alpha\beta} + 2\mu^2 B_{\alpha} B^{\alpha}] (16\pi)^{-1} (-g)^{1/2}\, d^4x - \sum_{i} e_i \int B_{\alpha}\, dx_i^{\alpha} \tag{3.4}$$

where $H_{\alpha\beta} = B_{\beta,\alpha} - B_{\alpha,\beta}$ and dx_i^{α} is the differential change in coordinate x^{α} along the worldline of particle i. Variation of S with respect to $g_{\alpha\beta}$ gives

$$T_{\alpha\beta} = [H_{\alpha}{}^{\gamma} H_{\beta\gamma} + \mu^2 B_{\alpha} B_{\beta} - \tfrac{1}{4} g_{\alpha\beta} (H_{\gamma\delta} H^{\gamma\delta} + 2\mu^2 B_{\gamma} B^{\gamma})] (4\pi)^{-1}, \tag{3.5}$$

so that for any unit timelike vector μ^{α}

$$\begin{aligned} T^{*}_{\alpha\beta} u^{\alpha} u^{\beta} = {} & [u^2 (B_{\alpha} u^{\alpha})^2 + H_{\alpha\beta} u^{\alpha} H_{\gamma}{}^{\beta} u^{\gamma} \\ & + \tfrac{1}{4} H_{\alpha\beta} H^{\alpha\beta}] (4\pi)^{-1} = [\mu^2 (B_{\alpha} u^{\alpha})^2 \\ & + \tfrac{1}{2} p^{\alpha\beta} H_{\gamma\alpha} u^{\gamma} H_{\delta\,\beta} u^{\delta} + \tfrac{1}{4} p^{\alpha\beta} p^{\gamma\delta} H_{\alpha\gamma} H_{\beta\delta}] (4\pi)^{-1} \end{aligned} \tag{3.6}$$

where we have introduced the positive-definite 3-metric $p^{\alpha\beta} = g^{\alpha\beta} + u^{\alpha} u^{\beta}$. We recall that any such metric can be (locally) brought to a form with positive diagonal components only. Then it is clear from (3.6) that $T^{*}_{\alpha\beta} u^{\alpha} u^{\beta}$ equals a sum of positive terms. Therefore, the vector field and the coupling satisfy the strong energy condition. If the stress energy for the particles itself satisfies the condition, then the mass of the static system composed of the above ingredients is always positive.

4. *Systems Disobeying the Strong Energy Condition*

Not all known forms of matter satisfy the strong energy condition. However, the general approach based on formula (3.2) may still remain meaningful. For example, suppose that for the u^{α} given by (3.1) $T^{*}_{\alpha\beta} u^{\alpha} u^{\beta}$ can be written as a positive definite quantity plus the 4-divergence of a vector field whose magnitude falls off faster than r^{-2} asymptotically. Then clearly the divergence can be integrated away and makes no contribution to the integral in (3.2). Thus M will be positive even though the strong energy condition is not necessarily satisfied.

An example of the above is provided by the case of a massive scalar field ϕ of Compton length m^{-1} coupled to point sources with strength f_i. The simplest parameter invariant action functional for field and coupling is

$$S = -\tfrac{1}{2}\int (\phi_{,\alpha}\phi_{,}{}^{\alpha} + m^2\phi^2)(-g)^{1/2}\, d^4x - \sum_i f_i \int \phi(-g_{\alpha\beta}\dot{x}_i{}^{\alpha}\dot{x}_i{}^{\beta})^{1/2}\, d\lambda_i \tag{4.1}$$

where $\dot{x}_i{}^{\alpha} = dx_i{}^{\alpha}/d\lambda_i$ for particle i, and λ_i is a parameter along its worldline. The wave equation derived by varying ϕ in (4.1) is

$$\phi_{,\alpha}{}^{;\alpha} - m^2\phi = \sum_i f_i(-g)^{1/2}\int \delta^4(x - x_i)(-g_{\alpha\beta}\dot{x}_i{}^{\alpha}\dot{x}_i{}^{\beta})^{1/2}\, d\lambda_i \tag{4.2}$$

and the stress energy is

$$\begin{aligned} T_{\alpha\beta} &= \phi_{,\alpha}\phi_{,\beta} - \tfrac{1}{2}g_{\alpha\beta}(\phi_{,\gamma}\phi_{,}{}^{\gamma} + m^2\phi^2) \\ &\quad - \sum_i f_i(-g)^{-1/2}\int \delta^4(x - x_i)\phi\dot{x}_i{}^{\alpha}\dot{x}_i{}^{\beta}(-g_{\gamma\delta}\dot{x}_i{}^{\gamma}\dot{x}_i{}^{\delta})^{-1/2}\, d\lambda_i \end{aligned} \tag{4.3}$$

We now find that for arbitrary u^{α}

$$\begin{aligned} T^*_{\alpha\beta}u^{\alpha}u^{\beta} &= (\phi_{,\alpha}u^{\alpha})^2 - \tfrac{1}{2}m^2\phi^2 + \sum_i f_i(-g)^{-1/2}\int \delta^4(x - x_i)\phi \\ &\quad \times [\tfrac{1}{2}(-g_{\alpha\beta}\dot{x}_i{}^{\alpha}\dot{x}_i{}^{\beta})^{1/2} - (-g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta})^{-1/2}(\dot{x}_i{}^{\gamma}u_{\gamma})^2]\, d\lambda_i \end{aligned} \tag{4.4}$$

whose sign is not evident.

Let us now choose $u^{\alpha} = \dot{x}^{\alpha}(-g_{\beta\gamma}\dot{x}^{\beta}\dot{x}^{\gamma})^{-1/2}$ at the positions of the particles, and in any continuous fashion elsewhere. Substituting this into (4.4), eliminating the integral term with the help of (4.2), and completing some derivatives we find

$$\begin{aligned} T^*_{\alpha\beta}u^{\alpha}u^{\beta} &= (\phi_{,\alpha}u^{\alpha})^2 + \tfrac{1}{2}\phi_{,\alpha}\phi_{,}{}^{\alpha} - \tfrac{1}{2}(\phi\phi_{,}{}^{\alpha})_{;\alpha} \\ &= \tfrac{1}{2}(\phi_{,\alpha}u^{\alpha})^2 + \tfrac{1}{2}p^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - \tfrac{1}{2}(\phi\phi_{,}{}^{\alpha})_{;\alpha} \end{aligned} \tag{4.5}$$

It is clear from this that $T^*_{\alpha\beta}u^{\alpha}u^{\beta}$ equals a sum of positive terms plus the perfect divergence of a vector that falls off as r^{-3} asymptotically. Furthermore, our choice of u^{α} is, in the static case (particles at rest), identical to the prescription (3.1). Hence, by our earlier reasoning the contribution of the scalar field plus coupling to the mass of a static system is positive. If, in addition, the matter itself satisfies the strong energy condition, then the mass will always be positive.

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